Microwave instability as a coherent light source

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We suggest that the coherent radiation observed recently at SURF II and the National Snychrotron Light Source vacuum ultraviolet ring is due to coherent microwave instability or, equivalently, to ''microbunching'' of the electron beams in the storage rings. We formulate in this paper the problem of microwave instability in the time domain. A linear homogeneous integro-differential equation for the perturbed current distribution is derived to describe the microwave coherent motion inside the electron bunch. For a specific band-limited high-frequency impedance, the equation can be diagonalized analytically and the eigensolution manifests explicitly the characteristics of microbunching. Coherent radiation power is also calculated for this solvable model, assuming the instability to be initiated by the shot noise inherent in the electron beam. [S1063-651X(98)10105-8]

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I. INTRODUCTION

We study in this paper the problem of coherent light emission from a storage ring due to a coherent instability of the circulating electron beam. We are particularly interested in the coherent instability associated with a local modulation of the beam charge density, namely, the ''microwave'' instability. A coherent instability is always associated with a loss of beam kinetic energy. If the cause of the instability is an evanescent impedance, for example, the rf cavity modes below the beam pipe cutoff frequency, then the energy lost by the beam is deposited into the impedance source, the rf cavity in this example. On the other hand, if the instability is caused by radiation impedance or, in other words, by the high-frequency component of the impedance such as the synchrotron radiation impedance [1] or the rf cavity parasitic modes above the beam pipe cutoff frequency, then the energy lost by the beam can be extracted from the ring as radiation.

It is the purpose of this paper to construct a solvable model for the microwave instability. We solve the initial value problem by assuming that the instability is started by the shot noise of the beam and then obtain the beam power loss. Above the beam pipe cutoff frequency, the beam power loss equals radiation power.

By the ''microwave instability'' we mean the instability in the region

$$\lambda \ll l_W \ll \sigma, \tag{1}$$

where λ is the perturbation wavelength (carrier wavelength), l_W is the wake length, and σ is the electron bunch length. All lengths are in units of radians. In terms of the carrier wave number $n_0 = 2\pi/\lambda$ and the impedance bandwidth $b \equiv (4\pi - l_W)/2l_W \cong 2\pi/l_W$, the above condition is equivalent to

$$n_0 \gg b \gg 2\pi/\sigma. \tag{2}$$

We shall refer to $2\pi/\sigma$ as the electron bunch bandwidth.

The microwave instability was discovered in 1975 by Boussard [2]. Boussard conjectured that the microwave-instability condition can be obtained simply by (i) writing down the well-known coasting beam instability condition [3] corresponding to n_0 and (ii) replacing the average beam current I_{av} that appears in the coasting beam condition by the peak current of the bunched beam. This conjecture was proved [4] in 1979 and the coherent mode of the microwave instability was later shown [5] to correspond to local modulation of the beam charge density inside the bunched beam, or to ''microbunching.''

Boussard's condition can be understood as follows. A coherent wave corresponding to the microwave instability is localized in a small region within the bunch as depicted in Fig. 1. Therefore, only the current density at the location of the coherent wave, not the average current, contributes to the instability condition.

With regard to the size of the coherent wave packet inside the electron bunch in Fig. 1, note that since the wake length l_W is the measure of the distance within which the wake field induced by an electron can affect another electron, the two electrons can maintain coherence with each other only if they are less than a wake length apart. As a consequence, the length of the coherent wave packet in Fig. 1 should be the same as the wake length. In other words, (coherence length) =(wake length).

Wang suggested [6] that the microwave instability is a source of coherent light above the beam-pipe cutoff frequency. Recently, coherent light emission from electron beams has been reported by two groups [8,9] working on two different storage rings; the wavelength of the coherent light was found to be much less than the bunch length in both

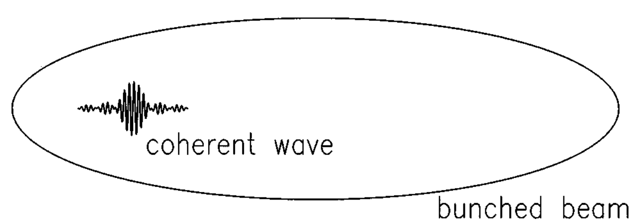

FIG. 1. Illustration of a coherent wave localized inside the electron bunch. The horizontal axis is ϕ and the vertical width of the oval at a given horizontal position ϕ represents the equilibrium-line density $\rho(\phi)$.

experiments. We suggest that the coherent light observed by these groups was due to the microwave instability.

This paper is organized as follows. In Sec. II we introduce the Vlasov equation appropriate to the microwave instability. We show that the Vlasov equation, together with an initial condition, is equivalent to a nonlinear integral equation. In Sec. III we linearize the integral equation to first order in the initial perturbation and show that the linearized integral equation is equivalent to a homogeneous linear integro-differential equation for the perturbed current distribution. We refer to this equation as the basic equation. The differential operator of the basic equation is second order in time.

In Sec. IV we introduce an orthonormal basis $\mathcal{B}_2$ of a finite-dimensional linear function space $\mathcal{M}$ in preparation for the discussion of a solvable model to be introduced in Sec. V. The space $\mathcal{M}$ is referred to as the modulation space and the shape of each member of the basis $\mathcal{B}_2$ looks like the envelope of the coherent wave packet of Fig. 1. Different members of $\mathcal{B}_2$ have identical shapes, but are located in different positions on the bunch. In Sec. V we introduce a model impedance and the corresponding basic equation for the current. We show that the basic equation can be diagonalized in terms of the orthonormal set of functions discussed in Sec IV, namely, in terms of the members of the basis. We also solve the initial value problem of the equation; the results are expressions of the perturbed current and the induced radiation field in terms of the initial perturbation current.

In Sec. VI we calculate the radiation power. We assume that the initial starting perturbation of the instability is due to the grainy characteristics of the shot noise in the beam. The graininess of the shot noise is averaged over in the expression for the radiation power.

The treatment of this paper is complimentary to that of Ref. [4]. While the earlier calculation was done primarily in the frequency domain, the calculation here is performed mainly in the time domain. Through solving a concrete model in the time domain, we attempt to make the characteristics of microbunching associated with microwave instability more transparent.

It is interesting to compare the microwave-instability-induced coherent light in a storage ring with the self-amplified spontaneous emission (SASE) [7] in a free-electron laser. In both cases, the radiation is a result of the longitudinal self-microbunching of the electron beam. The main difference is that in SASE, self-bunching is related to the electron transverse motion caused by the undulator, while in microwave instability, the microbunching is due to the longitudinal impedance alone and the transverse motion of the electron beam is not involved.

II. VLASOV EQUATION

A. Equation of motion and the Vlasov equation

We use the dynamical variables ϕ and ϵ to describe the beam particle motion

$$\phi=\theta-\omega_0 t, \quad \epsilon=E-E_0,$$

where θ describes the position around the ring and E_0 and ω_0 are, respectively, the nominal energy and the (angular) revolution frequency of the beam. It has been demonstrated [4] that the effect of synchrotron motion on the microwave instability is negligible since the growth rate of the microwave instability is much larger than the synchrotron frequency. Hence the equations of motion can be written as

$$\dot{\phi}=-\hat{\alpha}\epsilon, \quad \dot{\epsilon}=ce\mathcal{E}(\phi,t),$$

where $\mathcal{E}$ is the longitudinal electric field induced by the coherent signal of the beam and $\hat{\alpha}$ is related to the momentum compaction α by $\hat{\alpha}\equiv\alpha\omega_0/E_0$. The corresponding Vlasov equation is

$$\frac{\partial}{\partial t}\Psi(\phi,\epsilon,t)-\hat{\alpha}\epsilon\frac{\partial}{\partial\phi}\Psi+ce\mathcal{E}(\phi,t)\frac{\partial}{\partial\epsilon}\Psi=0, \tag{3}$$

where $\Psi(\phi,\epsilon,t)$ is the distribution function in (ϕ,ϵ) space. We are interested in the transient solution of this equation. So we next discuss the initial condition we impose on the Vlasov equation.

B. Initial condition and an integral representation

We assume that the coherent instability starts at $t=0$ and that the beam has no energy spread initially:

$$\Psi(\phi,\epsilon,t=0)=F(\phi)\delta(\epsilon).$$

Subject to this initial condition, the Vlasov equation (3) is equivalent to

$$\Psi(\phi,\epsilon,t)=F(\phi)\delta(\epsilon)-ce\int_0^t dt'\,\mathcal{E}(\phi+\hat{\alpha}\epsilon(t-t'),t')\times\frac{\partial}{\partial\epsilon}\Psi(\phi+\hat{\alpha}\epsilon(t-t'),\epsilon,t'). \tag{4}$$

This equation obviously satisfies the above-stated initial condition; it is straightforward to verify that Eq. (4) implies Eq. (3).

Note that Eq. (4) is not linear in Ψ since the induced field $\mathcal{E}$ depends on Ψ through the Maxwell equations. In this paper, the solution of the Maxwell equations will be represented by a generalized Ohm law in terms of the longitudinal beam impedance $Z_n(\omega)$. In what follows we shall deal with this integral equation form of the Vlasov equation.

C. Shot noise

The beam is composed of N particles. Denote the initial position of the jth particle by ϕ_j and represent the initial distribution by $F(\phi;\phi_1,\phi_2,\ldots,\phi_N)$. The granularity of the distribution due to the fact that the electrons are pointlike can be treated as shot noise. We assume that the partial coherence of the shot noise is entirely responsible for initiating the coherent instability.

The shot noise can be represented by

$$F(\phi;\phi_1,\phi_2,\ldots,\phi_N)=\frac{1}{N}\sum_{j=1}^{N}\delta(\phi-\phi_j). \tag{5}$$

Assuming that the probability density of each ϕ_j is $\rho(\phi_j)$ and averaging F over ϕ_j, we have

$$\langle F(\phi)\rangle \equiv \prod_{j=1}^{N} \int d\phi_j \rho(\phi_j) F(\phi;\phi_1,\phi_2,\ldots,\phi_N) = \rho(\phi). \tag{6}$$

$\rho(\phi)$ is normally referred to as the equilibrium-line density; here it is normalized to unity. We assume $\rho(\phi)$ to be a smooth function of ϕ, e.g., a Gaussian function with variance σ^2.

It is convenient to define the ''centered'' distribution function

$$f(\phi;\phi_1,\phi_2,\ldots,\phi_N) = F(\phi;\phi_1,\phi_2,\ldots,\phi_N) - \rho(\phi), \tag{7}$$

so that

$$\int d\phi f(\phi;\phi_1,\phi_2,\ldots,\phi_N) = 0$$

and

$$\langle f(\phi)\rangle = \prod_{j=1}^{N} \int d\phi_j \rho(\phi_j) f(\phi;\phi_1,\phi_2,\ldots,\phi_N) = 0.$$

III. AN INTEGRO-DIFFERENTIAL EQUATION FOR THE PERTURBED CURRENT

Let us denote by $I(\phi,t)$ the perturbed part of the electron beam current that is carrying on the coherent oscillation. We derive in this section a homogeneous linear integro-differential equation for $I(\phi,t)$ from Eq. (4). The induced coherent field $\mathcal{E}(\phi,t)$ can be expressed in terms of $I(\phi,t)$ by Ohm's law. We assume both $I(\phi,t)$ and $\mathcal{E}(\phi,t)$ to satisfy the conditions (1) and (2). We first derive an approximate form of Eq. (4).

A. Linearization

To linearize the integral equation (4), we iterate the equation once by substituting the first term on the right-hand side of the equation into the last term. In Sec. II C we split F into two parts: the equilibrium electron beam distribution ρ and the ''centered'' shot noise f. We regard ρ as the zeroth-order term and f the first-order term. We treat Eq. (4) up to $O(f)$.

Since $\mathcal{E}$ is a beam-induced field, it is linear in Ψ, namely, it is linear in both ρ and f. As mentioned in the Introduction, $\mathcal{E}$ consists, in our region of interest, of components with wavelength λ satisfying $\lambda \ll \sigma$, where σ is the bunch length of the smooth function ρ. However, the amount of such a small wavelength component of $\mathcal{E}$ induced by a smooth long bunch represented by ρ is negligible. In other words, the beam impedance responsible for microwave instability is nonvanishing only at very high frequency corresponding to n much greater than the bunch bandwidth $2\pi/\sigma$; therefore, the contribution of ρ to $\mathcal{E}$ is very small. We therefore ignore the ρ contribution to $\mathcal{E}$ and conclude that

$$\mathcal{E} = O(f).$$

(Suppose that the bunch bandwidth is smaller than the beam pipe cutoff and that there is evanescent beam impedance; then ignoring the ρ contribution to $\mathcal{E}$ is equivalent to ignoring the effect of the potential well distortion on the microwave instability.) For a similar reason, the term $F(\phi)\delta(\epsilon)$ can be replaced by $f(\phi)\delta(\epsilon)$ and thus the linearized integral equation can be written as

$$\Psi(\phi,\epsilon,t) = f(\phi)\delta(\epsilon) - ce\int_0^t dt' \mathcal{E}(\phi + \hat{\alpha}\epsilon(t-t'),t') \times \frac{\partial}{\partial\epsilon}\{\rho(\phi + \hat{\alpha}\epsilon(t-t'))\delta(\epsilon)\}. \tag{8}$$

Each term of this equation is $O(f)$.

B. Basic equation

The perturbed current is related to Ψ of Eq. (8) by

$$I(\phi,t) = eN\omega_0 \int d\epsilon \Psi(\phi,\epsilon,t).$$

Substituting Eq. (8) into this equation and then performing an integration by parts with respect to ϵ, we obtain

$$I(\phi,t) = 2\pi I_{av} f(\phi;\phi_1,\phi_2,\ldots,\phi_N) + 2\pi I_{av}\hat{\alpha}\rho(\phi) ce \int_0^t dt'(t-t')\frac{\partial}{\partial\phi}\mathcal{E}(\phi,t'), \tag{9}$$

where $I_{av} = eN\omega_0/2\pi$. Differentiating Eq. (9) with respect to t, we have

$$\frac{\partial}{\partial t} I(\phi,t) = 2\pi ce I_{av}\hat{\alpha}\rho(\phi)\int_0^t dt' \frac{\partial}{\partial\phi}\mathcal{E}(\phi,t') \tag{10}$$

and

$$\frac{\partial^2}{\partial t^2} I(\phi,t) = 2\pi ce I_{av}\hat{\alpha}\rho(\phi)\frac{\partial}{\partial\phi}\mathcal{E}(\phi,t). \tag{11}$$

From Eqs. (9) and (10) we have the initial conditions

$$I^{(0)}(\phi) \equiv I(\phi,t=0) = 2\pi I_{av} f(\phi;\phi_1,\phi_2,\ldots,\phi_N) \tag{12}$$

and

$$\dot{I}^{(0)}(\phi) \equiv \frac{\partial}{\partial t} I(\phi,t=0) = 0. \tag{13}$$

We note that Eq. (11) relates the current I to the beam-induced longitudinal electric field $\mathcal{E}$. In the remainder of this section we transform this equation into an equation that relates I to the impedance or, equivalently, to the wake field.

Let us Fourier transform the current I,

$$I(\phi,t) = \sum_n \int d\Omega\, I_n(\Omega)\exp(in\phi - i\Omega t) = \sum_n I_n(t)\exp(in\phi),$$

$$I_n(t)=\int d\Omega\; I_n(\Omega)\exp(-i\Omega t),$$

where the sum over n is from $-\infty$ to ∞. By definition of the longitudinal beam impedance $Z_n(\omega)$, the induced electric field is

$$\mathcal{E}(\phi,t)=-\frac{1}{2\pi R}\sum_n \int d\Omega\; I_n(\Omega)Z_n(n\omega_0+\Omega)\times\exp(in\phi-i\Omega t), \tag{14}$$

where R is the average ring radius. Recall that we are dealing with an impedance with a large bandwidth b [see condition (2)]; let us further assume that the bandwidth of Z is much larger than the bandwidth in Ω of $I_n(\Omega)$. Then, in the region of Ω where $I_n(\Omega)$ is appreciable, $Z_n(n\omega_0+\Omega)\cong Z_n(n\omega_0)$ and in terms of the shorthand notation $Z_n\equiv Z_n(n\omega_0)$ we have

$$\mathcal{E}(\phi,t)\cong-\frac{1}{2\pi R}\sum_n I_n(t)Z_n\exp(in\phi). \tag{15}$$

We now define the functions F and G by

$$F(\phi)=\frac{1}{2\pi}\sum_n Z_n\exp(-in\phi) \tag{16}$$

and

$$G(\phi)=F'(\phi)=\frac{1}{2\pi}\sum_n U_n\exp(-in\phi), \tag{17}$$

with $U_n\equiv-inZ_n$. The beam-induced electric field can now be written as

$$\mathcal{E}(\phi,t)=-\frac{1}{2\pi R}\int_{-\pi}^{\pi}d\phi' F(\phi'-\phi)I(\phi',t), \tag{18}$$

$$\frac{\partial}{\partial\phi}\mathcal{E}(\phi,t)=\frac{1}{2\pi R}\int_{-\pi}^{\pi}d\phi' G(\phi'-\phi)I(\phi',t) \tag{19}$$

and Eq. (11) becomes

$$\frac{\partial^2}{\partial t^2}I(\phi,t)=\kappa\int_{-\pi}^{\pi}d\phi' K(\phi,\phi')I(\phi',t), \tag{20}$$

with

$$\kappa=e\omega_0\hat{\alpha}I_{av}=e\alpha\omega_0^2I_{av}/E_0$$

and

$$K(\phi,\phi')=\rho(\phi)G(\phi'-\phi). \tag{21}$$

We shall refer to Eq. (20) with Eq. (21) as the basic equation.

We have to solve the basic equation subject to the initial conditions (12) and (13). For a coasting beam, $\rho(\phi)=1/2\pi=\mathrm{const}$, the right-hand side of Eq. (20) is a convolution integral; hence the equation can be diagonalized by a simple Fourier transform and the solution to the initial value problem is immediate. The mathematical complication for a bunched beam arises from the ϕ dependence of ρ, which causes different n's within the bunch bandwidth $1/\sigma$ to be coupled [4]. On the other hand, this coupling is also the reason why the eigenmode of the microwave instability is localized in the ϕ space even though the eigenmode of a coasting beam is well known to be of the form $\exp(in\phi)$ whose magnitude is a constant around the ring.

We have reduced in this section the problem of the microwave instability into the basic Eq. (20) with the kernel given by (21). In preparation for diagonalization of the operator (21) in a model to be introduced in Sec. V, we introduce in the next section a set of localized orthonormal functions.

IV. MODULATION SPACE AND MODULATION BASIS

The beam current distribution function $I(\phi,t)$ belongs to the infinite-dimensional space $\mathcal{S}$ of the functions of the variable $\phi, -\pi<\phi<\pi$. This space is spanned by the basis $\{\exp(in\phi)|n=0,\pm1,\pm2,\ldots,\pm\infty\}$. Define a $(2b+1)$-dimensional subspace $\mathcal{M}\subset\mathcal{S}$ that is spanned by the basis $\mathcal{B}_1=\{\exp(i\nu\phi)|\nu=0,\pm1,\pm2,\ldots,\pm b\}$.

We now introduce another orthonormal basis $\mathcal{B}_2\equiv\{\Gamma_\alpha\}$ of $\mathcal{M}$, which will be used in Sec. V to describe the modulation envelope of the coherent wave on the bunch. The integer b here will be identified with the impedance bandwidth b in the condition (2). In accordance with the condition, we assume b to be large.

Divide the storage ring circumference $-\pi<\phi\leqslant\pi$ into $2b+1$ equal parts with the lattice points

$$\phi=\phi_\alpha\equiv l_W\alpha/2,\quad(\alpha=0,\pm1,\pm2,\ldots,\pm b), \tag{22}$$

where

$$l_W=4\pi/(2b+1).$$

Definition. For $\alpha=0,\pm1,\pm2,\ldots,\pm b$,

$$\Gamma_\alpha(\phi)=[2\pi(2b+1)]^{-1/2}\sum_{\nu=-b}^{b}\exp[i\nu(\phi-\phi_\alpha)] \tag{23}$$

$$=[2\pi(2b+1)]^{-1/2}\frac{\sin[(b+1/2)(\phi-\phi_\alpha)]}{\sin[(\phi-\phi_\alpha)/2]}. \tag{24}$$

$\Gamma_\alpha(\phi)$ is a function peaked at $\phi=\phi_\alpha$ and the first zeros of the function are at $\phi=\phi_\alpha\pm l_W/2$. The functions Γ_α corresponding to different α's are of identical shape, but they are shifted from each other in ϕ by integer multiples of $l_W/2$. Neighboring Γ's have non-negligible overlaps; the peak of a Γ and one of the first zeros of the next Γ coincide. A few Γ's are depicted in Fig. 2. Note that if we take one of the $\Gamma_\alpha(\phi)$'s and multiply it by the carrier wave with wavelength λ satisfying Eq. (1), we obtain the coherent wave depicted in Fig. 1.

We refer to the space $\mathcal{M}$ as the modulation space, to the set $\mathcal{B}_2=\{\Gamma_\alpha\}$ as the modulation basis, and to the function $\Gamma_\alpha(\phi)$ as the modulation function. The following theorems about the modulation basis can be proven easily.

Theorem 1 (orthonormality).

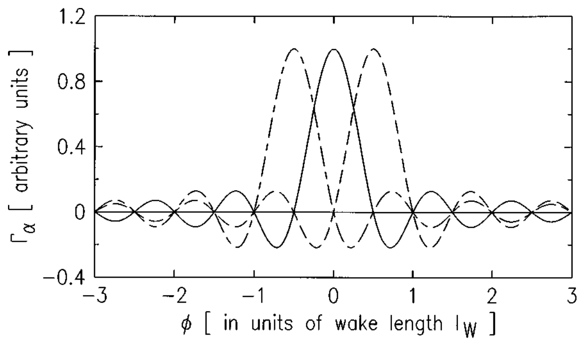

FIG. 2. Three neighboring modulation functions.

$$\int_{-\pi}^{\pi} d\phi\ \Gamma_\alpha(\phi)\Gamma_\beta(\phi)=\delta_{\alpha\beta}. \tag{25}$$

Theorem 2 (completeness in $\mathcal{M}$).

$$\sum_{\alpha=-b}^{b}\Gamma_\alpha(\phi)\Gamma_\alpha(\phi')=\frac{1}{2\pi}\sum_{\nu=-b}^{b}\exp[i\nu(\phi-\phi')]$$
$$=\sqrt{\frac{2b+1}{2\pi}}\Gamma_0(\phi-\phi'). \tag{26}$$

Theorem 2 implies that $\{\Gamma_\alpha\}$ spans the space $\mathcal{M}$ and that

$$\int_{-\pi}^{\pi} d\phi'\left[\sum_{\alpha=-b}^{b}\Gamma_\alpha(\phi)\Gamma_\alpha(\phi')\right]f(\phi')=f(\phi),\quad \forall f\in\mathcal{M}.$$

In other words, the operator (26) is the unity operator on $\mathcal{M}$. If f is orthogonal to $\mathcal{M}$, then the last integral vanishes.

We introduced in this section the modulation space $\mathcal{M}$ through the basis $\mathcal{B}_1$. We then introduced another basis $\mathcal{B}_2$ of the same space. The member Γ_α of $\mathcal{B}_2$ is localized in a small region of ϕ space, while the magnitude of the member $\exp(i\nu\phi)$ of $\mathcal{B}_1$ is distributed uniformly in the ϕ space. We will see that $\mathcal{B}_2$ is the set of the independent eigensolutions of the model to be introduced in the next section for the microwave instability.

V. SOLVABLE MODEL

In this section we introduce a model impedance that makes the operator (21) diagonalizable in terms of the modulation basis $\mathcal{B}_2=\{\Gamma_\alpha\}$ introduced in the preceding section. This makes the basic equation (20) solvable.

In what follows, the symbols n_0, b, λ, σ, and l_W carry the same meaning as in the Introduction. They are assumed to satisfy the conditions (1) and (2).

A. Model impedance and kernel

For $n>0$ the model impedance is defined by

$$U_n=\begin{cases}\bar{U} & \text{if } n_0-b\leqslant n\leqslant n_0+b\\ 0 & \text{otherwise.}\end{cases} \tag{27}$$

If we ignore terms of $O(b/n_0)$ relative to $O(1)$, Eq. (27) is equivalent to

$$Z_n=\begin{cases}\bar{Z} & \text{if } n_0-b\leqslant n\leqslant n_0+b\\ 0 & \text{otherwise,}\end{cases} \tag{28}$$

with $\bar{U}=-in_0\bar{Z}$. The impedances for $n<0$ can be obtained by using

$$U_{-n}=U_n^*,\quad Z_{-n}=Z_n^*,$$

where an $*$ indicates the complex conjugate.

We now calculate the corresponding model kernel $K(\phi,\phi')$. Substituting Eq. (27) into Eq. (17), we obtain

$$G(\phi)=[\bar{U}\exp(-in_0\phi)+\text{ c.c.}]\sqrt{\frac{2b+1}{2\pi}}\Gamma_0(\phi),$$

where Eq. (23), with $\alpha=0$ and the relation $U_{-n}=U_n^*$, has been used and c.c. stands for the complex conjugate. Combining the above equation with Eqs. (26) and (21), we have

$$K(\phi,\phi')=\{\bar{U}\exp[in_0(\phi-\phi')]+\text{ c.c.}\}\rho(\phi)$$
$$\times\sum_{\alpha=-b}^{b}\Gamma_\alpha(\phi)\Gamma_\alpha(\phi').$$

By assumption (2) and Eq. (24), $\rho(\phi)\cong\rho(\phi_\alpha)$ within the range of ϕ where $\Gamma_\alpha(\phi)$ is appreciable, namely, within the range $|\phi-\phi_\alpha|<\lambda_W/2$. Hence, introducing the notation

$$\rho_\alpha\equiv\rho(\phi_\alpha),$$

we can approximate

$$\rho(\phi)\Gamma_\alpha(\phi)\cong\rho_\alpha\Gamma_\alpha(\phi) \tag{29}$$

and

$$K(\phi,\phi')\cong\{\bar{U}\exp[in_0(\phi-\phi')]+\text{ c.c.}\}$$
$$\times\sum_{\alpha=-b}^{b}\rho_\alpha\Gamma_\alpha(\phi)\Gamma_\alpha(\phi').$$

Note that the kernel is now diagonalized in the modulation space. This is the form of the kernel we use below.

B. Diagonalization of the basic equation

We are now ready to solve the basic equation (20). First, let us collect here some of the relevant formulas obtained above:

$$\frac{\partial^2}{\partial t^2}I(\phi,t)=\kappa\int_{-\pi}^{\pi}d\phi' K(\phi,\phi')I(\phi',t), \tag{30}$$

$$K(\phi,\phi')=\{\bar{U}\exp[in_0(\phi-\phi')]+\text{c.c.}\}$$
$$\times\sum_{\alpha=-b}^{b}\rho_\alpha\Gamma_\alpha(\phi)\Gamma_\alpha(\phi'), \tag{31}$$

$$I^{(0)}(\phi)\equiv I(\phi,t=0)=2\pi I_{av}f(\phi;\phi_1,\phi_2,\ldots,\phi_N), \tag{32}$$

$$\dot{I}^{(0)}(\phi)\equiv\frac{\partial}{\partial t}I(\phi,t=0)=0, \tag{33}$$

$$I(\phi,t)=\sum_{n=-\infty}^{\infty}I_n(t)\exp(in\phi). \tag{34}$$

Decompose $I(\phi,t)$ into three mutually orthogonal components

$$I(\phi,t)=\exp(in_0\phi)J(\phi,t)+\exp(-in_0\phi)J^*(\phi,t)+\check{I}(\phi,t),$$

where $J\in\mathcal{M}$ ($\mathcal{M}$ was defined in Sec. IV.) Note that the first component is the contribution of $I_n(t)$ with $n\in[n_0-b,\ n_0+b]$, the second component is the contribution from $n\in[-n_0-b,\ -n_0+b]$, and $\check{I}$ is defined to be the contribution from n outside both these bands. These three components are orthogonal to each other since they belong to nonoverlapping bands of n. For example,

$$\int_{-\pi}^{\pi}d\phi\exp(in_0\phi)J(\phi,t)\check{I}^*(\phi,t)=0.$$

We observe right away from Eqs. (30) and (31) and the definition of $\check{I}$ above that

$$\frac{\partial^2}{\partial t^2}\check{I}(\phi,t)=0.$$

This, together with Eq. (33), implies that $\check{I}(\phi,t)$ is independent of t. We shall therefore ignore $\check{I}$ in the remainder of this paper and write

$$I(\phi,t)=\exp(in_0\phi)J(\phi,t)+\exp(-in_0\phi)J^*(\phi,t). \tag{35}$$

Note that in this equation the carrier wave $\exp(in_0\phi)$ is modulated by an element $J(\phi,t)$ of $\mathcal{M}$. Now using Eqs. (31) and (35), the basic equation (30) becomes a linear equation in the modulation space $\mathcal{M}$,

$$\frac{\partial^2}{\partial t^2}J(\phi,t)=\kappa\overline{U}\sum_{\alpha}\ \rho_\alpha\Gamma_\alpha(\phi)\int_{-\pi}^{\pi}d\phi'\Gamma_\alpha(\phi')J(\phi',t). \tag{36}$$

Since $\{\Gamma_\alpha\}$ is an orthonormal basis of $\mathcal{M}$, this equation is already diagonalized, the eigenfunction being $\Gamma_\alpha(\phi)$. Let

$$J(\phi,t)=\sum_{\alpha=-b}^{b}J_\alpha(t)\Gamma_\alpha(\phi); \tag{37}$$

then

$$\frac{d^2}{dt^2}J_\alpha(t)=\kappa\rho_\alpha\overline{U}J_\alpha(t). \tag{38}$$

The coherent frequency Ω_α of the mode α can readily be obtained from Eq. (38). Let

$$J_\alpha(t)\sim\exp(-i\Omega_\alpha t);$$

then

$$\Omega_\alpha^2=-\kappa\rho_\alpha\overline{U}. \tag{39}$$

Note that for each eigenvector $\Gamma_\alpha(\phi)$ of Eq. (36) there are actually two solutions for $J(\phi,t)$,

$$\Gamma_\alpha(\phi)\exp(-i\Omega_\alpha t),\quad \Gamma_\alpha(\phi)\exp(i\Omega_\alpha t). \tag{40}$$

This is a reflection of the fact that the basic equation (30) is second order in time.

Let us pause now and compare Eq. (39) with the corresponding result for a coasting beam. The coherent frequency Ω_n corresponding to the mode n of a coasting beam is given by [3]

$$\Omega_n^2=in\kappa Z_n/2\pi$$

and the corresponding eigenfunction is $\exp(in\phi)$. If we substitute $1/2\pi$ in this equation by ρ_α and replace inZ_n with $-\overline{U}$, we obtain Eq. (39). This amounts to a proof of the Boussard conjecture. It is worth repeating here, for emphasis, what was stated in the Introduction: The realization of the Boussard conjecture is a consequence of the coherent wave of the microwave instability being localized in the bunch, even though the coasting beam coherent wave, $\exp(in\phi)$, is spread throughout the whole ring.

C. Matching the initial condition

We have just found that corresponding to each mode number α there are two solutions (40) for $J(\phi,t)$. In order for these two solutions to satisfy the initial condition (33), they must combine to give

$$J(\phi,t)\sim\Gamma_\alpha(\phi)\cos\Omega_\alpha t.$$

Now adding up the contributions from all α, and including the contribution from the J^* term in Eq. (35), we obtain

$$I(\phi,t)=\sum_{\alpha=-b}^{b}\Gamma_\alpha(\phi)[I_\alpha^{(0)}\exp(in_0\phi)\cos\Omega_\alpha t+\text{ c.c.}], \tag{41}$$

where

$$I_\alpha^{(0)}=\int_{-\pi}^{\pi}d\phi\ \Gamma_\alpha(\phi)I^{(0)}(\phi)\exp(-in_0\phi). \tag{42}$$

We now calculate the longitudinal electric field $\mathcal{E}$ induced by $I(\phi,t)$. Combining the Eqs. (15), (28), and (41) and then using the relation $Z_{-n}=Z_n^*$, we obtain

$$\mathcal{E}(\phi,t)=-\left[\overline{Z}\sum_{\alpha=-b}^{b}\Gamma_\alpha(\phi)I_\alpha^{(0)}\times\exp(in_0\phi)\cos(\Omega_\alpha t)+\text{ c.c.}\right]/2\pi R. \tag{43}$$

We have thus succeeded in solving the transient problem of our model by expressing $I(\phi,t)$ and $\mathcal{E}(\phi,t)$ in terms of the initial current $I^{(0)}(\phi)$.

VI. RADIATION POWER

Having found the solution (41) and (43) to the initial value problem, we are ready to calculate the radiation power. The power lost by the beam (per radian of the beam distribution) is

$$P(\phi,t)=-R\mathcal{E}(\phi,t)I(\phi,t). \tag{44}$$

From conservation of energy, this is also the radiation power. The total radiation power is then

$$P_{tot}(t)=\int_{-\pi}^{\pi}d\phi P(\phi,t). \tag{45}$$

Substituting Eqs. (41) and (43) into Eq. (44) and ignoring the fast oscillating terms involving $\exp(\pm i2n_0\phi)$, we obtain

$$P(\phi,t)=\frac{1}{2\pi}\sum_{\alpha,\beta}[\overline{Z}I_\alpha^{(0)}I_\beta^{*(0)}\cos\Omega_\alpha t\,\cos\Omega_\beta^* t + \text{c.c.}]\Gamma_\alpha(\phi)\Gamma_\beta(\phi), \tag{46}$$

where both α and β are summed from $-b$ to b. Recall that $I^{(0)}$ is a messy collection of the grainy shot noise. We next statistically average Eq. (46) over the shot noise.

Averaging over shot noise

We use angular brackets, as we did in Sec. II C, to indicate averaging over the shot noise:

$$\langle P(\phi,t)\rangle=\frac{1}{2\pi}\sum_{\alpha,\beta}[\overline{Z}\langle I_\alpha^{(0)}I_\beta^{*(0)}\rangle\cos\Omega_\alpha t\,\cos\Omega_\beta^* t+\text{c.c.}]\times\Gamma_\alpha(\phi)\Gamma_\beta(\phi). \tag{47}$$

From Eq. (42)

$$\langle I_\alpha^{(0)}I_\beta^{*(0)}\rangle=\int_{-\pi}^{\pi}d\phi\int_{-\pi}^{\pi}d\phi'\exp[in_0(\phi'-\phi)]\times\Gamma_\alpha(\phi)\Gamma_\beta(\phi')\langle I^{(0)}(\phi)I^{(0)}(\phi')\rangle. \tag{48}$$

We must now evaluate $\langle I^{(0)}(\phi)I^{(0)}(\phi')\rangle$ or, equivalently, $\langle f(\phi)f(\phi')\rangle$ [cf. Eqs. (32), (5), and (7)]. The shot noise in a bunched beam is correlated because the bunch distribution function $\rho(\phi)$ is not uniform. In anticipation of the final results, we make the following remarks. Since $\Gamma_\alpha(\phi)$ is appreciable only within a width $l_W\cong 2\pi/b$ around $\phi=\phi_\alpha$, we see from Eq. (48) that we have to take the average only in this region. From the assumption $l_W\ll\sigma$, the bunch distribution $\rho(\phi)$ is nearly constant within the width l_W of $\Gamma_\alpha(\phi)$; we can therefore safely take the uniform shot noise average, assuming the noise density to be $N\rho_\alpha$.

We start from the definition

$$\langle f(\phi)f(\phi')\rangle=\int_{-\pi}^{\pi}\prod_{j=1}^{N}[\rho(\phi_j)d\phi_j]\times f(\phi;\phi_1,\phi_2,\ldots,\phi_N)\times f(\phi';\phi_1,\phi_2,\ldots,\phi_N).$$

Using Eqs. (5) and (7) on the above equation, we have

$$\langle f(\phi)f(\phi')\rangle=\frac{1}{N}[\rho(\phi)\delta(\phi'-\phi)-\rho(\phi)\rho(\phi')]. \tag{49}$$

The last term of this equation reflects the effects of the correlation induced by nonuniformity of the bunch. Let us ignore this term for now and show later that the contribution of this term is indeed negligible in our region of interest given by Eqs. (1) and (2).

If we ignore the last term of Eq. (49), then Eq. (48) yields

$$\langle I_\alpha^{(0)}I_\beta^{*(0)}\rangle=[(2\pi I_{av})^2/N]\int_{-\pi}^{\pi}d\phi\;\rho(\phi)\Gamma_\alpha(\phi)\Gamma_\beta(\phi).$$

With use of the approximation (29), the above equation becomes

$$\langle I_\alpha^{(0)}I_\beta^{*(0)}\rangle=[(eN\omega_0\rho_\alpha)^2/N\rho_\alpha]\delta_{\alpha,\beta}, \tag{50}$$

where Theorem 1 has been used. Substituting Eq. (50) into Eq. (47), we obtain

$$\langle P(\phi,t)\rangle=\sum_{\alpha=-b}^{b}\langle P_\alpha(\phi,t)\rangle, \tag{51}$$

with

$$\langle P_\alpha(\phi,t)\rangle=2\overline{\mathcal{R}}[(eN\omega_0\rho_\alpha)^2/2\pi N\rho_\alpha]|\cos\Omega_\alpha t|^2\Gamma_\alpha^2(\phi), \tag{52}$$

where $\overline{\mathcal{R}}$ is the resistive part of $\overline{Z}$. Had we assumed the initial beam to consist of uniform shot noise with density ρ_α we would have obtained the same result.

The calculation of averaged total radiation power from $\langle P(\phi,t)\rangle$ above is straightforward. The result is

$$\langle P_{\text{tot}}(t)\rangle=\sum_{\alpha=-b}^{b}\langle P_{\text{tot},\alpha}(t)\rangle, \tag{53}$$

with

$$\langle P_{\text{tot},\alpha}(t)\rangle=\int_{-\pi}^{\pi}d\phi\langle P_\alpha(\phi,t)\rangle=2\overline{\mathcal{R}}[(eN\omega_0\rho_\alpha)^2/2\pi N\rho_\alpha]|\cos\Omega_\alpha t|^2. \tag{54}$$

If we write

$$\Omega_\alpha=\Omega_{\alpha,R}+ig_\alpha, \tag{55}$$

where $\Omega_{\alpha,R}$ is the real coherent frequency shift and g_α is the growth rate of the mode α, then, for large t,

$$|\cos\Omega_\alpha t|^2 \cong \frac{1}{4}\exp(2g_\alpha t). \tag{56}$$

It is interesting to approximate the summation in Eq. (53) by an integral. If we set

$$\alpha \to \frac{2b+1}{2\pi}\phi \cong \frac{b}{\pi}\phi, \quad \sum_{\alpha=-b}^{b} \to \frac{b}{\pi}\int_{-\pi}^{\pi} d\phi,$$

$$\Omega_\alpha \to \Omega(\phi) = \Omega_R(\phi) + ig(\phi), \quad \rho_\alpha \to \rho(\phi),$$

then Eqs. (53) and (54) give an interesting expression

$$\langle P_{\mathrm{tot}}(t)\rangle = \frac{1}{2\pi}\overline{\mathcal{R}}\int_{-\pi}^{\pi} d\phi \frac{[eN\omega_0\rho(\phi)]^2}{N\rho(\phi)l_W}\exp[2g(\phi)t], \tag{57}$$

where Eq. (56) has been used.

We have so far ignored the effect of the last term of Eq. (49). Let us verify now that it is indeed ignorable. The last term of Eq. (49) adds to Eq. (50) a term proportional to

$$\frac{1}{N}\int_{-\pi}^{\pi} d\phi\; \rho(\phi)\Gamma_\alpha(\phi)\exp(-in_0\phi)$$
$$\times \int_{-\pi}^{\pi} d\phi'\rho(\phi')\Gamma_\beta(\phi')\exp(in_0\phi').$$

If we apply the approximation (29) to the integrands above, then both integrals vanish since $\exp(\pm in_0\phi)$ is orthogonal to $\Gamma_\alpha(\phi)$ and $\Gamma_\beta(\phi)$.

We constructed a model for microwave instability and calculated the radiation power under the assumption that the impedance is present only at frequencies above the beam pipe cutoff and that the initial condition for the coherent instability is the partially coherent signal from the ever-present shot noise in the beam. For our model impedance (27) and (28), the radiation power is given by Eq. (57). In the integrand of this equation, the denominator $N\rho(\phi)l_W$ is the number of particles in a coherence length or the number of electrons participating in a coherent mode. If the instability is not initiated by the shot noise but by some other mechanism, then the factor $1/N\rho(\phi)l_W$ would be replaced by something else.

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